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Semiclassical scar functions in phase space

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Abstract
We develop a semiclassical approximation for the scar function in the Weyl–Wigner representation in the neighborhood of a classically unstable periodic orbit of chaotic two-dimensional systems. The prediction of hyperbolic fringes, asymptotic to the stable and unstable manifolds, is verified computationally for a (linear) cat map, after the theory is adapted to a discrete phase space appropriate to a quantized torus. Characteristic fringe patterns can be distinguished even for quasi-energies where the fixed point is not Bohr-quantized. Also the patterns are highly localized in the neighborhood of the periodic orbit and along its stable and unstable manifolds without any long distance patterns that appear for the case of the spectral Wigner function.

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1. Introduction

The Gutzwiller trace formula provides a tool for the semiclassical evaluation of the energy spectrum in terms of the periodic orbits of the system. However, the number of long periodic orbits required to resolve the spectrum increases exponentially with the Heisenberg time T_H [1].

The semiclassical theory of short periodic orbits developed by Vergini and co-workers [2–7] is a formalism that allows us to obtain all the quantum information of a chaotic Hamiltonian system in terms of a very small number of short periodic orbits. In this context, the scar functions play a crucial role. These wavefunctions, that have been introduced in several previous works [7–12], live in the neighborhood of the classical trajectories, resembling the hyperbolic structure of the phase space in their immediate vicinity. This property makes them extremely suitable for investigating chaotic eigenfunctions. Recently, it has been shown that the matrix elements between scar functions provide information about the heteroclinic classical structure [6, 7].

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In order to perform further developments to this semiclassical theory of short periodic orbits, it is important to provide a general expression in phase space for the scar functions explicitly in terms of the classical invariants that generates the dynamics of the system. This is the purpose of this paper. In this context, the choice of the Weyl–Wigner representation provides a phase space vision which best allows a quantum classical comparison [13]. Also, for two degrees of freedom Hamiltonian systems the dynamics is studied entirely within a surface of section transversal to the periodic orbit. That is, the full dynamics is studied through a two-dimensional section map.

The semiclassical expression here deduced predicts characteristic hyperbolic patterns located in the neighborhood of the periodic orbits and along its stable and unstable manifolds. To show the validity of our approximation, we compare the general expression here founded with a 'realistic' system, the cat map, i.e. the quantization of linear symplectic maps on the torus. As was shown by Keating [23] in this case the semiclassical theory is exact, making these maps an ideal probe for our expression. After the formalism is adapted for a torus phase space, we see an important agreement between the semiclassical construction and the numerically computed Wigner scar functions for the cat maps. The characteristic hyperbolic patterns are clearly discernible even for non-Bohr-quantized values of the classical action.

Although most studies in phase space have adopted the Husimi representation [9, 12] this must be interpreted as a Gaussian smoothing of the Wigner function in an area of size $\hbar$ that usually dampens the fine structure extending along the stable and unstable manifolds [13]. This work is a first step to further objectives that includes the investigation of the matrix elements between scar functions.

Section 2 deals with the definition of scar functions and develops its relationship with the spectral operator. Then, the Weyl–Wigner representation of the scar function, here called the scar Wigner function, is studied. Mainly, its local hyperbolic form in the neighborhood of a single periodic orbit is obtained only in terms of classical objects. Also, this expression is compared to the already studied spectral Wigner function [14]. Although the formalism is valid for an autonomous Hamiltonian flux, we restrict the treatment to a surface of section transversal to the periodic orbit, in this way the flux is converted in a map on the section.

Section 3 is devoted to study the particular case of the cat map where not only the semiclassical theory is exact but also the linear approximation is valid throughout the torus. The scar Wigner function is then clearly visualized as a hyperbolic fringe pattern that is in agreement with the semiclassical expression derived in section 2. It has to be noted that we here choose values of N, the dimension of the Hilbert space, such that the spectrum of the cat map presents no degeneracies. In that case, the scar states are not the eigenfunctions of the propagator as Faure *et al* [12] show it happens. Also the fringe patterns here analyzed lie entirely within the scope of standard semiclassical theory, restricted to fairly short times. The interesting question concerning homoclinic recurrences depends on dynamics for longer times.

2. Scar Wigner functions

Scar function states are the main object of study of the current work. According to [7–12], the scar function $|\varphi_{X,\phi}\rangle$ of parameter ϕ constructed on a single periodic point $X = (P, Q)$ is defined as

$$|\varphi_{X,\phi}\rangle = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\phi t} f_T(t) \widehat{U}^t |X\rangle \tag{1}$$

where $T = \ln \hbar$ is the Ehrenfest time, $|X\rangle$ is a coherent state centered in the point X on the periodic orbit and $\widehat{U}^t$ is the unitary propagator that governs the quantum evolution of the system. While $f_T(t)$ is a decaying function that takes negligible values for $|t| > T/2$.

In [12], the function $f_T(t) = I_{[-T/2,T/2]}(t)$ that is simply the step function on the interval $[-T/2, T/2]$. While in [7] a cosine term is added in order to minimize the energy spreading, so that $f_T(t) = I_{[-T/2,T/2]}(t)\cos\left(\frac{\pi t}{T}\right)$. These wavefunctions have been shown, in the Husimi representation, to live in the neighborhood of the trajectory, resembling the hyperbolic structure of the phase space in their immediate vicinity [12].

It will be convenient for our purpose to choose $f_T(t) = \mathrm{e}^{-\left(\frac{4t}{T}\right)^2}$, that is the scar function is

$$|\varphi_{X,\phi}\rangle = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\phi t}\, \mathrm{e}^{(4t/T)^2}\, \widehat{U}^t |X\rangle. \tag{2}$$

On the other hand, the spectral operator (or energy Green function) is defined as

$$\widehat{G}_{E,\varepsilon} = \int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\mathrm{i}\phi t}\, \widehat{U}^t\, \mathrm{e}^{-t|\varepsilon|/\hbar} \tag{3}$$

with $\widehat{U}^t = \mathrm{e}^{\frac{-\mathrm{i}}{\hbar}\widehat{H}t}$ and $\phi = \frac{E}{\hbar}$ so that

$$\widehat{G}_{E,\varepsilon} = \frac{1}{\pi\hbar}\int_{-\infty}^{\infty} \mathrm{d}t\, \mathrm{e}^{\frac{\mathrm{i}}{\hbar}(E-\widehat{H})t}\, \mathrm{e}^{-t|\varepsilon|/\hbar} = -\frac{1}{\pi}\,\mathrm{Im}\left(\frac{1}{\widehat{H} - E - \mathrm{i}\varepsilon}\right) \tag{4}$$

$$= \frac{1}{\pi}\frac{\varepsilon}{(\widehat{H} - E)^2 + \varepsilon^2} \equiv \delta_\varepsilon(E - \widehat{H}) = \sum_n |\psi_n\rangle\langle\psi_n|\delta_\varepsilon(E - E_n) \tag{5}$$

where $|\psi_n\rangle$ are the Hamiltonian eigenfunctions with energy E_n and $\delta_\varepsilon(x)$ is a normalized function whose width ε can be taken to be arbitrary small, so that δ_ε tend to the Dirac δ function as $\varepsilon \to 0$. Then $\widehat{G}_{E,\varepsilon}$ is not a pure state but a statistical mixture. In the limit $\varepsilon \to 0$, the spectral operator is a comb of delta functions on the eigenangles, whose amplitudes are the corresponding individual density matrices $|\psi_n\rangle\langle\psi_n|$. For values of ε larger than the mean level spacing, several eigenstates contribute to the spectral operator in a Lorentzian-like smoothing of energy width ε.

As we mentioned earlier, for the function $f_T(t)$ in (1) we can choose any damping term with characteristic time T as, for example, $\mathrm{e}^{-\frac{t|\varepsilon|}{\hbar}}$ with $\varepsilon \approx \hbar/T$. So that, it is possible to establish a relationship of the scar functions with the spectral operator

$$|\varphi_{X,\phi}\rangle \cong \widehat{G}_{\hbar\phi,\hbar/T}|X\rangle, \tag{6}$$

that is the scar function is the spectral operator acting on the coherent state centered at the point X on the periodic orbit.

The purpose of this work is to study the scar function in phase space by means of the Weyl–Wigner representation $\rho_{X,\phi}(x)$, here called the scar Wigner function,

$$\rho_{X,\phi}(x) = \mathrm{tr}[\widehat{R}_x\widehat{\rho}_{X,\phi}] \tag{7}$$

where $\widehat{\rho}_{X,\phi} \equiv |\varphi_{X,\phi}\rangle\langle\varphi_{X,\phi}|$ is the density matrix of the scar function and $\widehat{R}_x$ are the set of reflection operators through the points $x = (p, q)$ in phase space [15, 16]. The Weyl–Wigner representation, defined through the set of reflection operators (see the appendix), has the advantage to show structures of size lower than $\hbar$ [17] while the Husimi representation is a Gaussian smoothing on a region of size $\hbar$ [13, 18] and hence has a lower resolution.

For the case of the spectral operator its Wigner function

$$G_{E,\varepsilon}(x) = W(E, \varepsilon, x) = \mathrm{tr}[\widehat{R}_x\widehat{G}_{E,\varepsilon}] \tag{8}$$

known as the spectral Wigner function, originally introduced by Berry [19], has been recently shown to present important scarring features [14] some of which would be present in the scar Wigner function $\rho_{X,\phi}(x)$ as will be seen in this work.

It must be noted that by construction

$$\widehat{\rho}_{X,\phi} \equiv |\varphi_{X,\phi}\rangle\langle\varphi_{X,\phi}| \tag{9}$$

is a pure state while $\widehat{G}_{E,\varepsilon}$ is a statistical average but both are related by

$$\widehat{\rho}_{X,\phi} \cong \widehat{G}_{\hbar\phi,\hbar/T}|X\rangle\langle X|\widehat{G}^{\dagger}_{\hbar\phi,\hbar/T}.$$

Inserting the definition of scar function (1) with (9) into (7) it can be seen that

$$\rho_{X,\phi}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}t'\,\mathrm{d}t\,\mathrm{e}^{\mathrm{i}\phi t} f_T(t) f_T(t')\langle X|\widehat{U}^{-t'}\widehat{R}_x\widehat{U}^{t}|X\rangle. \tag{10}$$

Let us now use the decomposition of the propagator in terms of reflection operators [15]

$$\widehat{U}^{t} = \left(\frac{1}{\pi\hbar}\right)^{L}\int \mathrm{d}x\, U^{t}(x)\widehat{R}_x \tag{11}$$

where $\int \mathrm{d}x$ is an integral over the whole phase space and L is the number of degrees of freedom. The scar Wigner function is then expressed as

$$\rho_{X,\phi}(x) = \left(\frac{1}{\pi\hbar}\right)^{2L}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \mathrm{d}t'\,\mathrm{d}t\, f_T(t) f_T(t')\exp[\mathrm{i}\phi(t-t')]$$
$$\times\int \mathrm{d}x_1\int \mathrm{d}x_2 U^{-t'}(x_2)U^{t}(x_1)\langle X|\widehat{R}_{x_2}\widehat{R}_x\widehat{R}_{x_1}|X\rangle. \tag{12}$$

The coherent states on the points $X = (p, q)$ in phase space are obtained by translating to X the ground state of the harmonic oscillator, its position representation is

$$\langle q|X\rangle = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}}\exp\left[-\frac{\omega}{2\hbar}(q-Q)^2 + \mathrm{i}\frac{P}{\hbar}\left(q-\frac{Q}{2}\right)\right]. \tag{13}$$

For simplicity, unit frequency ($\omega = 1$) and mass ($m = 1$) are chosen for the harmonic oscillator without loss of generality. The overlap of two coherent states is then

$$\langle X|X'\rangle = \exp\left[-\frac{(X-X')^2}{4\hbar} - \frac{\mathrm{i}}{2\hbar}X\wedge X'\right], \tag{14}$$

where the wedge product

$$X\wedge X' = PQ' - QP' = (\mathcal{J}X)\cdot X',$$

the second equation also defines the symplectic matrix $\mathcal{J}$. As is shown in the appendix, the action of the reflection operator $\widehat{R}_x$ on a coherent state $|X\rangle$ is the x reflected coherent state

$$\widehat{R}_x|X\rangle = \exp\left[\frac{\mathrm{i}}{\hbar}X\wedge x\right]|2x - X\rangle \tag{15}$$

and the product of three reflections also gives a reflection

$$\widehat{R}_{x_2}\widehat{R}_x\widehat{R}_{x_1} = \exp\left[\frac{\mathrm{i}}{\hbar}\Delta_3(x_2, x_1, x)\right]\widehat{R}_{x_R(x_2,x_1,x)} \tag{16}$$

where

$$\Delta_3(x_2, x_1, x) = 2(x_2 - x)\wedge(x_1 - x) \tag{17}$$

is the area of the oriented triangle whose sides are centered on the points x_2, x_1 and x, respectively, while

$$x_R(x_2, x_1, x) = x_2 - x + x_1 \tag{18}$$

is the point vertex formed by the sides of the triangle centered on the points x_2 and x_1 (see figure 1).

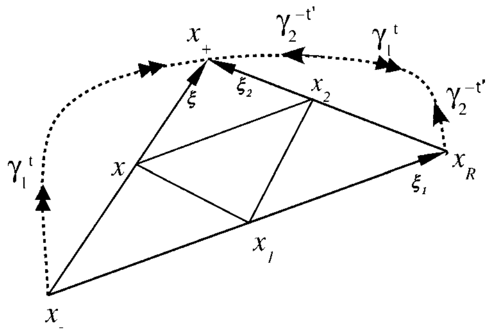

Figure 1. In order to correctly compose the canonical transformations the end point of the orbit γ_1^t, centered in x_1, and the initial point of $\gamma_2^{-t'}$, centered in x_2, must coincide in x_R. The resulting classical orbit $\gamma^{t-t'}$ that joins the points x_- and x_+ and has its center point in x is the composition of the orbits γ_1^t and $\gamma_2^{-t'}$.

Performing the product (16) applied to a coherent state, as in (15), and then overlapping with (14) it is shown that

$$\langle X|\widehat{R}_{x_2}\widehat{R}_x\widehat{R}_{x_1}|X\rangle = \mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Delta_3(x_2,x_1,x)}\exp -\left[\frac{(X-x_2+x-x_1)^2}{\hbar}\right]. \tag{19}$$

The result (19) is inserted into (12) in order to perform the double phase space integrals

$$\begin{aligned} I &= \left(\frac{1}{\pi\hbar}\right)^{2L}\int \mathrm{d}x_1\int \mathrm{d}x_2 U^{-t'}(x_2)U^{t}(x_1)\langle X|\widehat{R}_{x_2}\widehat{R}_x\widehat{R}_{x_1}|X\rangle \\ &= \left(\frac{1}{\pi\hbar}\right)^{2L}\int \mathrm{d}x_1\int \mathrm{d}x_2 U^{-t'}(x_2)U^{t}(x_1)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Delta_3(x_2,x_1,x)}\exp\left[-\frac{(X-x_2+x-x_1)^2}{\hbar}\right]. \end{aligned} \tag{20}$$

If the damping exponential term is omitted in (20), the double phase space integral is simply performed using the product of symbols in phase space [15] that is

$$U^{t-t'}(x) = \widehat{U}^{-t'}\widehat{U}^{t}(x) = \left(\frac{1}{\pi\hbar}\right)^{2L}\int \mathrm{d}x_1\int \mathrm{d}x_2 U^{-t'}(x_2)U^{t}(x_1)\,\mathrm{e}^{\frac{\mathrm{i}}{\hbar}\Delta_3(x_2,x_1,x)} \tag{21}$$

but the presence of the exponential term has to be kept into account so that

$$I = U^{t-t'}(x)\exp\left[-\frac{[X-x_R(x)]^2}{\hbar}\right] \tag{22}$$

where now $x_R(x)$ is a point in phase space that only depends on x (and of course on the times t and t') since its dependence on x_1 and x_2 has been integrated. Another approach to obtain this result is to perform in (20) the semiclassical approximation for the propagator

$$U_{sc}^t(x) = \sum_\gamma \frac{\mathrm{e}^{\mathrm{i}\alpha_\gamma^t}}{\left|\det M_\gamma^t + 1\right|^{\frac{1}{2}}}\exp\left[\frac{\mathrm{i}}{\hbar}S_\gamma^t(x)\right] \tag{23}$$

where the summation is over all the classical orbits γ whose center lies on the point x after having evolved a time t [15]. Then $S_\gamma^t(x)$ is the classical center-generating function of the orbit, from which the chord ξ joining the initial and final points of the orbit is obtained:

$$\xi = -\mathcal{J}\frac{\partial S_\gamma^t(x)}{\partial x}, \tag{24}$$

while $M_\gamma^t = \frac{\partial^2 S_\gamma^t(x)}{\partial x^2}$ stand for the monodromy matrix and α_j^t its Maslov index.

Inserting the semiclassical propagator (23) into the phase space integral (20), we get

$$I = \left(\frac{1}{\pi\hbar}\right)^{2L} \sum_{\gamma_1}\sum_{\gamma_2} \exp\left[\mathrm{i}\left(\alpha^{t}_{\gamma_1} + \alpha^{-t'}_{\gamma_2}\right)\right] \int\int \frac{\exp\left[-\frac{1}{\hbar}(X - x_2 + x - x_1)^2\right]}{\left|\det M^{-t'}_{\gamma_2} + 1\right|^{\frac{1}{2}}\left|\det M^{t}_{\gamma_1} + 1\right|^{\frac{1}{2}}} \times \exp\left\{\frac{\mathrm{i}}{\hbar}\left[S^{t}_{\gamma_1}(x_1) - S^{t'}_{\gamma_2}(x_2) + \Delta_3(x_2, x_1, x)\right]\right\} \mathrm{d}x_1\,\mathrm{d}x_2, \tag{25}$$

where the summations are performed over orbits γ_1 and γ_2 whose centers lie on the points x_1 and x_2 after having evolved time t and $-t'$, respectively. While for the phase space integrals in (25) we use the stationary phase approximation. That is, the term inside the integral is relevant only in the neighborhoods of the points x_1 and x_2 for which the phase in the highly oscillating term is stationary. In our case, the phase $S(x_2, x_1, x) = S^{t}_{\gamma_1}(x_1) - S^{t'}_{\gamma_2}(x_2) + \Delta_3(x_2, x_1, x)$ is stationary only for the points $x_2(x), x_1(x)$ that fulfil simultaneously the conditions

$$\begin{aligned} \frac{\partial S}{\partial x_1} &= \frac{\partial S^{t}_{\gamma_1}}{\partial x_1} + \frac{\partial \Delta_3}{\partial x_1} = \mathcal{J}\xi_1 + 2\mathcal{J}(x_2 - x) = 0, \\ \frac{\partial S}{\partial x_2} &= -\frac{\partial S^{t'}_{\gamma_2}}{\partial x_2} + \frac{\partial \Delta_3}{\partial x_2} = -\mathcal{J}\xi_2 - 2\mathcal{J}(x_1 - x) = 0. \end{aligned} \tag{26}$$

That is, $x_2(x), x_1(x)$ must be such that the end point of the chord $\xi_1 = 2(x_2 - x)$, generated by the orbit γ^{t}_{1}, centered in x_1 coincides with the initial point of the chord $\xi_2 = 2(x - x_1)$, generated by the orbit $\gamma^{-t'}_{2}$, centered in x_2. Note that the matching point is exactly $x_1 + \frac{\xi_1}{2} = x_2 - \frac{\xi_2}{2} = x_1 + x_2 - x = x_R$. In this way, we can compose both canonical transformations γ^{t}_{1} and $\gamma^{-t'}_{2}$, so as to obtain the canonical transformation $\gamma^{t-t'}$ whose center-generating function is

$$S^{t-t'}_{\gamma}(x) = S^{t}_{\gamma_1}(x_1) - S^{t'}_{\gamma_2}(x_2) + \Delta_3(x_2, x_1, x) \tag{27}$$

such that

$$\xi = -\mathcal{J}\frac{\partial S^{t-t'}_{\gamma}(x)}{\partial x} = \xi_1 + \xi_2.$$

For clarity this situation is depicted in figure 1 (see also [15]). In that figure, x_- denotes the initial point of the orbit γ^{t}_{1}, while its time t evolution

$$\mathcal{L}^{t}(x_-) = x_R(x). \tag{28}$$

Whereas x_+ stand for a time $t - t'$ evolution

$$\mathcal{L}^{(t-t')}(x_-) = x_+.$$

Also x is the center point of x_- and x_+ . That is,

$$x = \frac{x_- + x_+}{2} = \frac{x_- + \mathcal{L}^{(t-t')}(x_-)}{2}. \tag{29}$$

Once the stationary phase points have been determined, the amplitude terms in the first line in (25) must be evaluated on these stationary points, while the phase must be expanded in there neighborhoods up to second order in x_1 and x_2 and then perform the resulting Gaussian integral. Adopting this procedure it results in

$$I = \sum_{\gamma} \exp\left(\mathrm{i}\alpha^{t-t'}_{\gamma}\right) \frac{\exp \frac{\mathrm{i}}{\hbar}\left[S^{t-t'}_{\gamma}(x)\right]}{\left|\det M^{t-t'}_{\gamma} + 1\right|^{\frac{1}{2}}} \exp\left[-\frac{(X - x_R(x))^2}{\hbar}\right], \tag{30}$$

where $x_R(x) = x_2(x) - x + x_1(x)$ with the points $x_1(x)$ and $x_2(x)$ fulfilling (26) so that $x_R(x)$ corresponds to the end point of the orbit γ^{t}_{1} or to the initial point of $\gamma^{-t'}_{2}$ as shown in figure 1, while the orbit $\gamma^{t-t'}$ is the composition of γ^{t}_{1} and $\gamma^{-t'}_{2}$.

Now the stationary-phase-integrated result (30) is inserted into the scar Wigner function (12) to obtain its semiclassical approximation:

$$\rho^{SC}_{X,\phi}(x) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dt'\, dt\, f_T(t)\, f_T(t')\exp[i\phi(t-t')] \times \sum_{\gamma} \exp\left(i\alpha_{\gamma}^{t-t'}\right) \frac{\exp \frac{i}{\hbar}\left[S_{\gamma}^{t-t'}(x)\right]}{\left|\det M_{\gamma}^{t-t'} + 1\right|^{\frac{1}{2}}} \exp\left[-\frac{(X - x_R(x))^2}{\hbar}\right]. \tag{31}$$

With the choice for $f_T(t)$ made in (2) and performing the change of variables $t_1 = t - t'$ and $t_2 = t + t'$ it is possible to separate part of the time integrals so that

$$\rho^{SC}_{X,\phi}(x) = 2\int_{-\infty}^{\infty}\left\{\sum_{\gamma}\frac{e^{-4t_1^2/T^2}}{\left|\det M_{\gamma}^{t_1} + 1\right|^{\frac{1}{2}}}\exp\left[i\left(\frac{1}{\hbar}S_{\gamma}^{t_1}(x) + t_1\phi + \alpha_{\gamma}^{t_1}\right)\right] \times \int_{-\infty}^{\infty} e^{-4t_2^2/T^2}\exp\left[-\frac{(X - x_R(x,t_1,t_2))^2}{\hbar}\right] dt_2\right\} dt_1. \tag{32}$$

In the last expression we have explicitly written the time dependence of the point x_R. Recall that the summation is performed over all the classical orbits γ^{t_1} that after a time t_1 have their center point lying in the phase space point x, while $x_R(x, t_1, t_2)$ is the end point after the same trajectory has spent a time $t = (t_1 + t_2)/2$.

So as to study $\rho^{SC}_{X,\phi}(x)$ in the neighborhood of the point X on a periodic orbit, the main contribution in the sum over classical orbits in (32) will come from the particular orbit that lies the closest to the periodic orbit passing through X. Other orbits contributions will be highly damped by the exponential term involving $X - x_R$. Then, only this particular orbit will be taken into account.

Also, a stationary phase treatment of (32) involves a time integral in t_1 with the phase

$$S_{\gamma}^{\phi}(x) = S_{\gamma}^{t_1}(x) + t_1\phi + \alpha_{\gamma}^{t_1}. \tag{33}$$

The same stationary phase treatment appears for the spectral Wigner function and has been already studied by Berry [20] and Ozorio de Almeida [15]. In that case, stationary phase methods show that both the energy shell and the periodic orbit are Airy caustics that separate an oscillatory behavior from an evanescent one obtaining maximum amplitude close to the caustic. Also, there are phase oscillations along the orbit while not on the energy shell. In addition, for the case here studied (32) the damping term in $X - x_R(x)$ implies that the structure of the scar function is highly localized in the neighborhood of the periodic orbit passing through X. That is, on the Airy caustic.

In order to study the phase space structure of the scar function where it is relevant and to manifestly show the hyperbolic structure of the instable periodic orbit, we will study the scar function on a surface of section that is transversal to the flux and passing through X. In analogy with classical Poincare surfaces of section. The flux restricted to this section is now a map on the section, for this map the time is discrete and time integrals must be replaced by summations.

The study of autonomous fluxes through a map on surface of section is a standard procedure, in the case of billiards this is done through the well-known Birkhoff coordinates. Also, the quantum surface of section methods is shown to be exact [21] for general Hamiltonian systems.

For this procedure, we can choose coordinates near the periodic orbit of period τ such that one coordinate is the energy E and the conjugate coordinate is the time along the orbit.

With this choice of coordinates, a point $x = (\tilde{x}, t, E)$ with now $\tilde{x}$ being a $(2L - 2)$ vector on the so-called central surface of section [15]. In order to perform our study on this surface of section near the periodic point X, we also linearize the flux in the neighborhood of the orbit through X. That is, $x_+ = \mathcal{L}_\gamma^{t_1}(x_-) \approx M_\gamma^{t_1} x_-$, where $M_\gamma^{t_1}$ is the symplectic matrix denoting this linearized time evolution. As was shown in [15], in the transformation $x_+ = M_\gamma^{t_1} x_-$ for times t_1 that are integer multiples of τ, $t_1 = n_1\tau$, the points $x_+ = (\tilde{x}_+, t_+, E_+)$ and $x_- = (\tilde{x}_-, t_-, E_-)$ on the surface of section have the same energy $(E_+ = E_-)$ and time along the orbit $(t_+ = t_-)$ so we can write

$$M_\gamma^{t_1} = \left(\begin{array}{c|c} m_\gamma^{t_1} & 0 \\ \hline 0 & \begin{array}{c} 10 \\ 01 \end{array} \end{array}\right) \tag{34}$$

with

$$\det\left[1 + M_\gamma^{t_1}\right] = 4\det\left[1 + m_\gamma^{t_1}\right], \tag{35}$$

where $m_\gamma^{t_1}$ is now the $(2L - 2) \times (2L - 2)$ symplectic matrix for the center map determined by the orbit γ on the surface section, that is

$$\tilde{x}_+ = m_\gamma^{t_1} \tilde{x}_-. \tag{36}$$

From now on, the $2L$-dimensional autonomous flux is studied through the $2L - 2$ map on the mentioned surface of section. Also the point X on the periodic orbit of the flux is a periodic point for the map on the section.

Let us define on the section $x' = \tilde{x} - X$. In the same way, $\tilde{x}_-$ the initial point of the classical orbit γ^{t_1} can be written as $\tilde{x}_- = X + \delta_-$ so that its time t linearized evolution on the section (t is again an integer multiple of τ) is

$$x_R = m_\gamma^t \tilde{x}_- = m_\gamma^t\,(X + \delta_-) = X + m_\gamma^t \delta_-. \tag{37}$$

Note that for the fixed point $m_\gamma^t(X) = X$. Also for the center point defined in (29)

$$\tilde{x} = X + x' = \frac{\tilde{x}_- + m_\gamma^{(t-t')}\tilde{x}_-}{2} = \frac{X + \delta_- + X + m_\gamma^{t-t'}\delta_-}{2} = X + \left(m_\gamma^{t-t'} + 1\right)\frac{\delta_-}{2}. \tag{38}$$

Inverting this last expression we see that

$$\delta_- = 2\left(m_\gamma^{t-t'} + 1\right)^{-1} x' \tag{39}$$

that is inserted into (37) to obtain

$$x_R = X + 2\frac{m_\gamma^t}{\left(m_\gamma^{t-t'} + 1\right)} x' \tag{40}$$

and finally

$$X - x_R = -2\frac{m_\gamma^t}{\left(m_\gamma^{t-t'} + 1\right)} x'. \tag{41}$$

For the case of a map with one degree of freedom (corresponding to a two degrees of freedom flux), the eigenvalues of the symplectic matrix m_γ^t are $\exp(-\lambda t)$ and $\exp(\lambda t)$ (λ is the stability exponent of the orbit) corresponding to the stable and unstable directions generated by the vectors $\vec{\xi}_s$ and $\vec{\xi}_u$, respectively. Let us define q' and p' as canonical coordinates along the stable and unstable directions, respectively, such that $x' = (p', q') = q'\vec{\xi}_s + p'\vec{\xi}_u$, with $\vec{\xi}_u \wedge \vec{\xi}_s = 1$. Then, using (41) with the diagonal representation of the symplectic matrix, the scalar product $(X - x_R)^2 = (X - x_R) \cdot (X - x_R)$ takes the form

$$(X - x_R)^2 = \frac{1}{\cosh^2\left(\lambda\frac{t-t'}{2}\right)}\left[p'^2\,\mathrm{e}^{\lambda(t+t')}\xi_u^2 + q'^2\,\mathrm{e}^{-\lambda(t+t')}\xi_s^2 + 2p'q'\vec{\xi}_u.\vec{\xi}_s\right] \tag{42}$$

where $\xi_u^2 = \vec{\xi}_u \cdot \vec{\xi}_u$ and $\xi_s^2 = \vec{\xi}_s \cdot \vec{\xi}_s$. It can also be easily shown that

$$\left|\det m_\gamma^t + 1\right|^{\frac{1}{2}} = 2\cosh\left(\frac{\lambda t}{2}\right). \tag{43}$$

The center action for the orbit that is close to the fixed point (periodic orbit for the flux) is given by

$$S_\gamma^t(\tilde{x}) = tS_X + x' B_\gamma^t x' + O(x'^3) \tag{44}$$

where S_X is the action of the fixed point for which the Maslov index $\alpha_\gamma^t = t\alpha_\gamma$. Let us define the action $\widetilde{S}_X = S_X + \hbar\alpha_\gamma$ in order to include the Maslov index in the action. B^t is the symmetric matrix such that

$$\mathcal{J} B_\gamma^t = \frac{1 - m_\gamma^t}{1 - m_\gamma^t} \tag{45}$$

with

$$\mathcal{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. \tag{46}$$

Thus, B_γ^t is the Cayley parameterization of m_γ^t. Using the stable and unstable directions as coordinate axes, the result is that

$$\mathcal{J} B_\gamma^t = \begin{bmatrix} \tanh(t\lambda/2) & 0 \\ 0 & -\tanh(t\lambda/2) \end{bmatrix}. \tag{47}$$

Inserting (47) into action (44) and putting together with (43) and (42) into expression (32) for the values of the time that are integer multiples of τ and for the points $\tilde{x}$ on the surface of section, we obtain

$$\rho_{X,\phi}^{\mathrm{SC}}(\tilde{x}) = \sum_{n_1=-\infty}^{\infty} \left\{ \exp\left[\mathrm{i}\left(\phi + \frac{\widetilde{S}_X}{\hbar}\right) t_1 - \frac{2\mathrm{i}}{\hbar} p'q' \tanh\left(\frac{t_1\lambda}{2}\right)\right] \frac{\mathrm{e}^{-4t_1^2/T^2}}{2\cosh\left(\frac{t_1\lambda}{2}\right)} \right.$$
$$\left. \times \sum_{n_2=-\infty}^{\infty} \mathrm{e}^{-4t_2^2/T^2} \exp\left[-\frac{1}{\hbar\cosh^2\left(\lambda\frac{t_1}{2}\right)} \left(p'^2 \mathrm{e}^{\lambda t_2}\xi_u^2 + q'^2 \mathrm{e}^{-\lambda t_2}\xi_s^2 + 2p'q'\vec{\xi}_u \cdot \vec{\xi}_s \right) \right] \right\}, \tag{48}$$

where $t_1 = n_1\tau$ and $t_2 = n_2\tau$ with n_1 and n_2 integer numbers. This last expression represents the semiclassical scar Wigner function on the surface of section that cuts transversally the periodic orbit on the point X. Note that all the other orbits contributions in (32) were neglected. It is important to see that (48) shows that the dependence of the scar Wigner function on the phase space variables has two aspects, a phase oscillating term only depending on the product $p'q'$ and a damping term. The phase oscillating terms show phase coherence along the stable and unstable directions where the damping factor is equal to one. Phase coherence also holds where the product $p'q'$ is constant, i.e. along each successive hyperbola that has the stable and unstable manifolds as asymptotes. This implies in phase oscillations across the hyperbolae the amplitude of the oscillations decreases with increasing λ and will be maximal for $\phi = \frac{E}{\hbar}$ corresponding to the Bohr-quantized orbit, while the damping term implies that away from the asymptotes the amplitude of the phase oscillations presents a Gaussian decreasing with increasing the distance.

This amplitude decaying is not present in the Spectral Wigner function that has strong oscillations away from the periodic orbit and its stable and unstable manifolds [14]. This fact implies important superpositions of the hyperbolic patterns from different orbits that usually wash out all the structure.

In order to deal with the double infinite time summations in (48) we perform a cutoff for the values of $|t_1|$ and $|t_2|$ greater than T, the Ehrenfest time, beyond which the time-dependent Gaussian became negligible. Remember also the discussion according to the choice of the function $f_T(t)$ in (1).

3. Scar Wigner functions for the cat map

Now the present theory is applied to the cat map, i.e. the linear automorphism on the 2-torus generated by the 2×2 symplectic matrix $\mathcal{M}$, that takes a point x_- to a point x_+ : $x_+ = \mathcal{M}x_-\text{mod}(1)$. In other words, there exists an integer two-dimensional vector $\mathbf{m}$ such that $x_+ = \mathcal{M}x_- - \mathbf{m}$. Equivalently, the map can also be studied in terms of the center-generating function [24]. This is defined in terms of center points

$$x \equiv \frac{x_+ + x_-}{2} \tag{49}$$

and chords

$$\xi \equiv x_+ - x_- = -\mathcal{J}\frac{\partial S(x, \mathbf{m})}{\partial x}, \tag{50}$$

where

$$S(x, \mathbf{m}) = xBx + x(B - \mathcal{J})\mathbf{m} + \tfrac{1}{4}\mathbf{m}(B + \widetilde{\mathcal{J}})\mathbf{m} \tag{51}$$

is the center-generating function. Here, B is a symmetric matrix (the Cayley parameterization of $\mathcal{M}$, as in (47)), while

$$\widetilde{\mathcal{J}} = \left[\begin{array}{c|c} 0 & 1 \\ \hline 1 & 0 \end{array}\right]. \tag{52}$$

We will study here the cat map with the symplectic matrix

$$\mathcal{M} = \begin{bmatrix} 2 & 3 \\ 1 & 2 \end{bmatrix} \qquad \text{and} \qquad \text{symmetric matrix } B = \begin{bmatrix} -\frac{1}{3} & 0 \\ 0 & 1 \end{bmatrix}. \tag{53}$$

This map is known to be chaotic (ergodic and mixing) as all its periodic orbits are hyperbolic. The periodic points x_l of integer period l are labeled by the winding numbers $\mathbf{m}$, so that

$$x_l = \begin{pmatrix} p_l \\ q_l \end{pmatrix} = (\mathcal{M}^l - 1)^{-1}\mathbf{m}. \tag{54}$$

The first periodic points of the map are the fixed points at $(0, 0)$ and $\left(\frac{1}{2}, \frac{1}{2}\right)$ and the periodic orbits of period 2 are $\left[\left(0, \frac{1}{2}\right), \left(\frac{1}{2}, 0\right)\right]$, $\left[\left(\frac{1}{2}, \frac{1}{6}\right), \left(\frac{1}{2}, \frac{5}{6}\right)\right]$, $\left[\left(0, \frac{1}{6}\right), \left(\frac{1}{2}, \frac{2}{6}\right)\right]$, $\left[\left(0, \frac{5}{6}\right), \left(\frac{1}{2}, \frac{4}{6}\right)\right]$ and $\left[\left(0, \frac{2}{6}\right), \left(0, \frac{4}{6}\right)\right]$. The eigenvalues of $\mathcal{M}$ are $\mathrm{e}^{-\lambda}$ and e^{λ} with $\lambda = \ln(2 + \sqrt{3}) \approx 1.317$. This is then the stability exponent for the fixed points, whereas the exponents must be doubled for orbits of period 2. All the eigenvectors have directions $\vec{\xi}_s = \left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$ and $\vec{\xi}_u = \left(1, \frac{1}{\sqrt{3}}\right)$ corresponding to the stable and unstable directions, respectively.

Quantum mechanics on the torus implies a finite Hilbert space of dimension $N = \frac{1}{2\pi\hbar}$ and that positions and momenta are defined to have discrete values in a lattice of separation $\frac{1}{N}$ [16, 22]. The cat map was originally quantized by Hannay and Berry [22] in the coordinate representation the propagator is

$$\langle \mathbf{q}_k | \hat{\mathbf{U}}_{\mathcal{M}} | \mathbf{q}_j \rangle = \left(\frac{\mathrm{i}}{N}\right)^{\frac{1}{2}} \exp\left[\frac{\mathrm{i}2\pi}{N}(k^2 - jk + j^2)\right], \tag{55}$$

where the states $\langle q|\mathbf{q}_j\rangle$ are periodic combs of Dirac delta distributions at positions $q = j/N \bmod(1)$, with j being an integer in $[0, N-1]$. In the Weyl representation [16], the quantum map has been obtained in [24] as

$$\mathbf{U}_{\mathcal{M}}(x) = \frac{2}{|\det(\mathcal{M}+1)|^{\frac{1}{2}}} \sum_{\mathbf{m}} \mathrm{e}^{\mathrm{i}2\pi N[S(x,\mathbf{m})]}$$
$$= \frac{2}{|\det(\mathcal{M}+1)|^{\frac{1}{2}}} \sum_{\mathbf{m}} \mathrm{e}^{\mathrm{i}2\pi N[xBx + x(B-\mathcal{J})\mathbf{m} + \frac{1}{4}\mathbf{m}(B+\widetilde{\mathcal{J}})\mathbf{m}]}, \tag{56}$$

where the center points are represented by $x = \left(\frac{a}{N}, \frac{b}{N}\right)$ with a and b being the integer numbers in $[0, N-1]$ for odd values of N [16]. There exists an alternative definition of the torus Wigner function which also holds for even N. However, it is constructed on the quarter torus and this compactification scrambles the hyperbolic patterns.

The fact that the symplectic matrix $\mathcal{M}$ has equal diagonal elements implies the time-reversal symmetry and then the symmetric matrix B has no off-diagonal elements. This property will be valid for all the powers of the map and, using (56), we can see that it implies in the quantum symmetry

$$\mathbf{U}^l_{\mathcal{M}}(p,q) = \left(\mathbf{U}^l_{\mathcal{M}}(-p,q)\right)^* = \left(\mathbf{U}^l_{\mathcal{M}}(p,-q)\right)^*, \tag{57}$$

for any integer value of l.

It has been shown [22] that the unitary propagator is periodic (nilpotent) in the sense that for any value of N there is an integer $k(N)$ such that

$$\hat{\mathbf{U}}^{k(N)}_{\mathcal{M}} = \mathrm{e}^{\mathrm{i}\phi}. \tag{58}$$

Hence, the eigenvalues of the map lie on the $k(N)$ possible sites:

$$\left\{\exp\left[\frac{\mathrm{i}(2m\pi+\phi)}{k(N)}\right]\right\}, \qquad 1 \leqslant m \leqslant k(N). \tag{59}$$

For the cases where $k(N) < N$ there are degeneracies and the spectrum does not behave as expected for chaotic quantum systems. In spite of the peculiarities in this map, a very weak nonlinear perturbation of cat maps restores the universal behavior of nondegenerate chaotic quantum systems spectra [25]. Eckhardt [26] has argued that typically the eigenfunctions of cat maps are random.

The scar Wigner function on the torus depends on the definition of the periodic coherent state [27, 28] with $\langle p\rangle = P$ and $\langle q\rangle = Q$. In accordance to (13)

$$\langle \mathbf{X}|\mathbf{q}_k\rangle = \sum_{j=-\infty}^{\infty} \exp\left\{2\pi N\left[-\frac{(j+Q-k/N)^2}{2\omega^2} - \mathrm{i}P\left(j+\frac{Q}{2}-k/N\right)\right]\right\}. \tag{60}$$

The scar function is then defined on the torus as

$$|\varphi_{\mathbf{X},\phi}\rangle = \sum_{t=-\infty}^{\infty} \mathrm{e}^{\mathrm{i}\phi t}\,\mathrm{e}^{-(4t/T)^2}\mathbf{U}^t_{\mathcal{M}}|\mathbf{X}\rangle. \tag{61}$$

Remember that for maps, time only takes discrete values, then the time integral in (2) has been in this case replaced by a summation. Also, as we have already discussed, for our numerical computations we truncate the sum for times $|t| > T/2$ where the Gaussian damping term became negligible. Now, the scar Wigner function on the torus is

$$\rho_{\mathbf{X},\phi}(x) = \mathbf{Tr}[\widehat{\mathbf{R}}_x|\varphi_{\mathbf{X},\phi}\rangle\langle\varphi_{\mathbf{X},\phi}|] \tag{62}$$

where the trace is now taken on torus Hilbert space and $\widehat{\mathbf{R}}_x$ are the periodic reflection operators on the torus [16]. In order to construct the semiclassical scar Wigner functions on the torus we have to periodize the construction. This is done merely using the recipe [16] that for any operator its Weyl representation on the torus $\mathbf{A}(x)$ is obtained from its analog in the plane $A(x)$ by

$$\mathbf{A}(x) = \sum_{j=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} (-1)^{2ja+2kb+jkN} A\left(x + \frac{(k, j)}{2}\right). \tag{63}$$

This leads to the property

$$\mathbf{A}\left(x + \frac{(k, j)}{2}\right) = (-1)^{2ja+2kb+jkN} \mathbf{A}(x), \tag{64}$$

for the torus Weyl symbol. In the case of the scar Wigner function, the phase factor in (64) leads to four images of the scar pattern, supported by the integer lattice. The images centered on (P, Q), $\left(P + \frac{1}{2}, Q\right)$ and $\left(P, Q + \frac{1}{2}\right)$ are all identical, whereas $\left(P + \frac{1}{2}, Q + \frac{1}{2}\right)$ centers a pattern which is the negative of the other ones. This fact has already been studied for the Wigner function of coherent states [14].

In figure 2, we compare the exact scar Wigner function for a cat map with $N = 223$, a value for which the quantum map has no degeneracies, in (*a*) with the semiclassical approximation, correspondingly for $\hbar = 1/(2\pi N)$, in (*d*). The case studied corresponds to the periodic point at $(1/2, 1/2)$ whose action is $S_X = 0.75$ for a value of ϕ that does not Bohr-quantize the orbit. Figures 2(*b*) and (*c*) show, respectively, the horizontal and vertical sections in the neighborhood of the periodic point of the objects plotted in figures 2(*a*) and (*d*).

It can easily be observed that the four images of the scar patterns are present both for the exact and semiclassical scar Wigner functions. Also, the semiclassical approximation properly describes the overall behavior of the exact dynamical system with detail of oscillations of the order 0.02.

Although deviations are present, particularly seen in figure 2(*b*), they are due to the contribution of longer orbits to (48). In particular, the period-2 orbit on the points $\left[\left(0, \frac{2}{6}\right)\left(0, \frac{4}{6}\right)\right]$ has $(1/2, 1/2)$ as center point. These periodic orbits only have contributions for the adequate values of t_1, in the specified case t_1 must be odd for the period-2 orbit to contribute.

4. Discussion

As was already observed for the case of the spectral Wigner function, the imprint of the classical hyperbolicity on the scar Wigner function is so clear that it can be detected even for quasi-energies that do not correspond to a Bohr-quantized periodic orbit.

The general features exhibited by our calculations should also be discernible for nonlinear systems as, indeed, our deduction was not restricted to cat maps. Though the theory in section 2 is only local, we conjecture that distorted hyperbolae asymptotic to curved stable and unstable manifolds will bear fringes reaching out from the periodic point. In the case of a chaotic Hamiltonian for two degrees of freedom, this pattern should emerge in two-dimensional sections cutting the periodic orbit at a point. This plane should be transverse to that of the orbit and in the energy shell.

The cutoff with time in our definition of the scar Wigner function affords equal treatment to all periodic orbits in the denominator of (48). The reason why the main contribution comes from only one hyperbolic orbit is that for an orbit of period n to have coherent contribution in (48) $t - t'$ must have specific values that differ in n. That is, the number of significant terms in (48) is divided by n. Also, T increases only logarithmically with $\hbar$ so that the semiclassical limit $\lambda \to n\lambda$ implies in a cutoff for longer orbits.

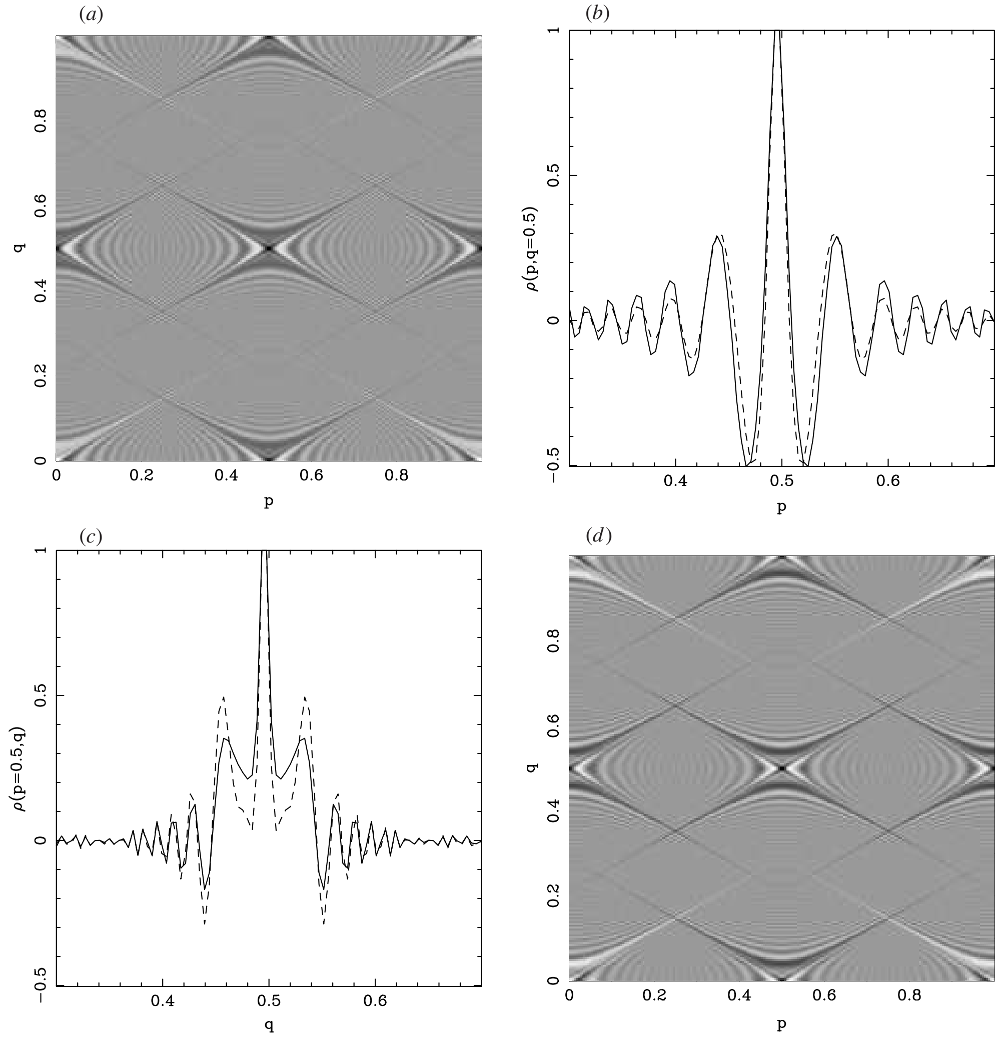

Figure 2. Scar Wigner function with $N = 223$ constructed on the fixed point $(0, 0)$ for a non-Bohr-quantized value of ϕ. The black colors represent large positive values of the Wigner function while large negative values are represented in white. (*a*) Exact result for the cat map. (*b*) We compare sections of the exact and the semiclassical scar Wigner functions near the periodic point for $q = 0.5$ (horizontal section). The solid line represents the section of the exact scar Wigner function, while by the dashed lines we show the semiclassical approximation. (*c*) Same as (*b*) but for $p = 0.5$ (vertical section). (*d*) Semiclassical approximation $\rho_{\mathbf{X},\phi}{}^{\mathrm{SC}}(x)$.

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Appendix A. Reflection operators in phase space

Among the several representations of quantum mechanics, the Weyl–Wigner representation is the one that performs a decomposition of the operators that act on the Hilbert space on the basis formed by the set of unitary reflection operators. In this appendix, we review the definition and some properties of these reflection operators.

First of all we construct the family of unitary operators

$$\hat{T}_q = \exp(-\mathrm{i}\hbar^{-1} q \cdot \hat{p}), \qquad \hat{T}_p = \exp(\mathrm{i}\hbar^{-1} p \cdot \hat{q}), \tag{A.1}$$

and following [15] we define the operator corresponding to a general translation in phase space by $\xi = (p, q)$ as

$$\hat{T}_\xi \equiv \exp\left(\frac{\mathrm{i}}{\hbar}\xi \wedge \hat{x}\right) \equiv \exp\left[\frac{\mathrm{i}}{\hbar}(p \cdot \hat{q} - q \cdot \hat{p})\right]$$
$$= \hat{T}_p \hat{T}_q \exp\left[-\frac{\mathrm{i}}{2\hbar} p \cdot q\right] = \hat{T}_q \hat{T}_p \exp\left[\frac{\mathrm{i}}{2\hbar} p \cdot q\right], \tag{A.2}$$

where naturally $\hat{x} = (\hat{p}, \hat{q})$. In other words, the order of $\hat{T}_p$ and $\hat{T}_q$ affects only the overall phase of the product, allowing us to define the translation as above. $\hat{T}_\xi$ is also known as a *Heisenberg operator*. Acting on the Hilbert space we have

$$\widehat{T}_\xi |q_a\rangle = \exp\left[\frac{\mathrm{i}}{\hbar} p \left(q_a + \frac{q}{2}\right)\right] |q_a + q\rangle \tag{A.3}$$

and

$$\widehat{T}_\xi |p_a\rangle = \mathrm{e}^{-\frac{\mathrm{i}}{\hbar} q (p_a + \frac{p}{2})} |p_a + p\rangle. \tag{A.4}$$

We, hence, verify their interpretation as translation operators in phase space. The group property is maintained within a phase factor:

$$\hat{T}_{\xi_2} \hat{T}_{\xi_1} = \hat{T}_{\xi_1 + \xi_2} \exp\left[\frac{-\mathrm{i}}{2\hbar} \xi_1 \wedge \xi_2\right] = \hat{T}_{\xi_1 + \xi_2} \exp\left[\frac{-\mathrm{i}}{\hbar} D_3(\xi_1, \xi_2)\right], \tag{A.5}$$

where D_3 is the symplectic area of the triangle determined by two of its sides. Evidently, the inverse of the unitary operator $\hat{T}_\xi^{-1} = \hat{T}_\xi^\dagger = \hat{T}_{-\xi}$.

The set of operators corresponding to phase space reflections $\hat{R}_x$ about the points $x = (p, q)$ in phase space is formally defined in [15] as the Fourier transform of the translation (or Heisenberg) operators

$$\widehat{R}_x \equiv (4\pi\hbar)^{-L} \int \mathrm{d}\xi \exp\left[\frac{\mathrm{i}}{\hbar} x \wedge \xi\right] \widehat{T}_\xi. \tag{A.6}$$

Their action on the coordinate and momentum bases are

$$\hat{R}_x |q_a\rangle = \mathrm{e}^{2\mathrm{i}(q - q_a)p/\hbar} |2q - q_a\rangle \tag{A.7}$$
$$\hat{R}_x |p_a\rangle = \mathrm{e}^{2\mathrm{i}(p - p_a)q/\hbar} |2p - p_a\rangle, \tag{A.8}$$

displaying the interpretation of these operators as reflections in phase space. Also, using the coordinate representation of the coherent state (13) and the action of reflection on the coordinate basis (A.7), we can see that the action of the reflection operator $\widehat{R}_x$ on a coherent state $|X\rangle$ is the x reflected coherent state

$$\widehat{R}_x |X\rangle = \exp\left(\frac{\mathrm{i}}{\hbar} X \wedge x\right) |2x - X\rangle. \tag{A.9}$$

This family of operators has the property that they are a decomposition of the unity (completeness relation):

$$\hat{1} = \frac{1}{2\pi\hbar} \int \mathrm{d}x \, \hat{R}_x, \tag{A.10}$$

and also they are orthogonal in the sense that

$$\mathrm{Tr}\left[\hat{R}_{x_1}\hat{R}_{x_2}\right] = 2\pi\hbar\delta(x_2 - x_1). \tag{A.11}$$

Hence, an operator $\hat{A}$ can be decomposed in terms of reflection operators as follows:

$$\hat{A} = \frac{1}{2\pi\hbar}\int \mathrm{d}x\, A_W(x)\hat{R}_x. \tag{A.12}$$

With this decomposition, the operator $\hat{A}$ is mapped on a function $A_W(x)$ lying in phase space, the so-called Weyl–Wigner symbol of the operator. Using (A.11) it is easy to show that $A_W(x)$ can be obtained by performing the following trace operation:

$$A_W(x) = \mathrm{Tr}[\hat{R}_x\hat{A}].$$

Of course, as it is shown in [15], the Weyl symbol also takes the usual expression in terms of matrix elements of $\hat{A}$ in coordinate representation

$$A_W(x) = \int \left\langle q - \frac{Q}{2}|\hat{A}|q + \frac{Q}{2}\right\rangle \exp\left(-\frac{\mathrm{i}}{\hbar}pQ\right)\mathrm{d}Q.$$

It was also shown in [15] that reflection and translation operators have the following composition properties:

$$\widehat{R}_x\widehat{T}_\xi = \widehat{R}_{x-\xi/2}\exp\left[-\frac{\mathrm{i}}{\hbar}x\wedge\xi\right], \tag{A.13}$$

$$\widehat{T}_\xi\widehat{R}_x = \widehat{R}_{x+\xi/2}\exp\left[-\frac{\mathrm{i}}{\hbar}x\wedge\xi\right], \tag{A.14}$$

$$\widehat{R}_{x_1}\widehat{R}_{x_2} = \widehat{T}_{2(x_2-x_1)}\exp\left[\frac{\mathrm{i}}{\hbar}2x_1\wedge x_2\right] \tag{A.15}$$

so that

$$\widehat{R}_x\widehat{R}_x = \widehat{1}. \tag{A.16}$$

Now using (A.15) and (A.14) we can compose three reflections so that

$$\widehat{R}_{x_2}\widehat{R}_x\widehat{R}_{x_1} = \exp\left[\frac{\mathrm{i}}{\hbar}\Delta_3(x_2, x_1, x)\right]\widehat{R}_{x_2-x+x_1} \tag{A.17}$$

where $\Delta_3(x_2, x_1, x) = 2(x_2 - x)\wedge(x_1 - x)$ is the area of the oriented triangle whose sides are centered on the points x_2, x_1 and x, respectively (see figure 1).

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